

**PARABOLIC GEOMETRIES
FOR PEOPLE THAT LIKE PICTURES**

**LECTURE 2: GEOMETRY AS
“LIE THEORY WITH EXTRA STEPS”
(PART 1)**

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Recall from last time that we can think of $I(2)$, the Lie group of Euclidean isometries over \mathbb{R}^2 , as the space of configurations for ourselves as pedestrians on the Euclidean plane. Equivalently, we can think of it as the orthonormal frame bundle over \mathbb{R}^2 , with each $\phi \in I(2)$ identified with the orthonormal frame $\phi_* : T_0\mathbb{R}^2 \approx \mathbb{R}^2 \rightarrow T_{\phi(0)}\mathbb{R}^2$ given by the pushforward of ϕ at 0 and with bundle map given by the natural quotient map $q_{O(2)} : I(2) \rightarrow \mathbb{R}^2 \cong I(2)/O(2)$, $\phi \mapsto \phi(0)$. On $I(2)$, we have the Maurer-Cartan form $\omega_{I(2)}$, which gives a kind of canonical (co)framing to $I(2)$ that agrees with how we typically describe motion, letting us speak about “forward velocity” without having to specify our current configuration. We learned how to move inside of a Lie group, and now it is time to use this knowledge to do some geometry.

The goal of this lecture is to practice thinking in terms of the Maurer-Cartan form, and in particular, to start demonstrating why we can think of geometry as “Lie theory with extra steps”. Along the way, we will happen to accomplish the following as well:

- Rephrase some basic aspects of Euclidean geometry in terms of $I(2)$
- Notice the interpretation of the adjoint representation coming from our intuition for conjugation
- Prove basic results in Euclidean geometry using isometries
- Think about how the structure of $I(2)$ determines Euclidean geometry

By the end of this lecture, we should be able to talk about how the geometric structure of the Euclidean plane comes from the Lie group $I(2)$ acting transitively on it. In the next lecture, we will begin to explore geometries determined by other Lie groups, along the same vein. We should note that, while elements of this lecture show up later, the important take-away from this is that the pair $(I(2), O(2))$ determines Euclidean geometry; if you already know this with some detail, then it can be safely skipped.

1. “ISN’T GEOMETRY ABOUT CIRCLES AND LINES AND STUFF?”

In the previous lecture, we used lines and other notions of Euclidean geometry to determine the form of the isometry group $I(2)$. Now, we would like to try going the other way; starting from $I(2)$, we will describe Euclidean geometry.

As we mentioned at the end of the first lecture, when we “walk in a straight line”, this really just corresponds to moving with constant translational velocity—which is to say, translational velocity that is constant with respect to the Maurer-Cartan form. Thus, a line will just be the full path of such motion projected onto the plane.

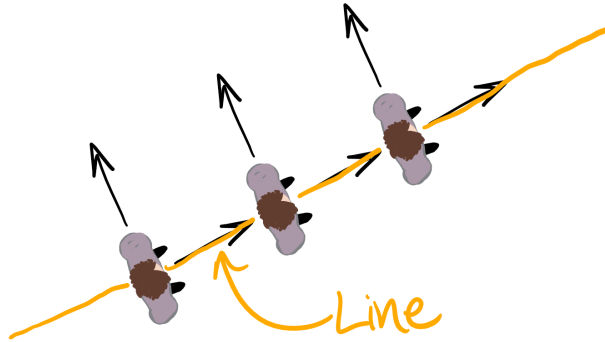


FIGURE 1. A line is the projection to the plane of a curve with $\omega_{I(2)}$ -constant translational velocity and zero angular velocity

Definition 1.1. A *line* on the Euclidean plane $I(2)/O(2) \cong \mathbb{R}^2$ is a subset of the form $q_{O(2)}(g \exp(\mathbb{R}v))$ for some $g \in I(2)$ and some nonzero $v \in \mathbb{R}^2 < \mathfrak{i}(2)$.

Choosing to use *right*-translations to define lines might seem odd to the uninitiated. Indeed, if you are not already familiar with Cartan geometries, then it probably seems easier to define lines as orbits of one-parameter subgroups of translations acting from the *left* on the Euclidean plane. Unfortunately, in this case, the reason for using right-translations is somewhat obscured by the fact that the subgroup of translations is normal¹ in $I(2)$. By the end of the next lecture, the reason for this choice will be obvious, but for now, we will just say that we always want to be able to *move* along lines (and, later, geodesics and other distinguished curves).

¹Recall that $N \leq G$ is *normal* in G if and only if $gN = Ng$ for all $g \in G$. In other words, when N is normal, every motion R_n with $n \in N$ has a corresponding transformation $L_{n'}$ for some $n' \in N$ that behaves the same way at a given $g \in G$.

Note that, for every $x \in \mathbb{R}^2$, we have $\tau_x \in q_{O(2)}^{-1}(x)$. Thus, if $x + \mathbb{R}v$ is a line in the usual sense on \mathbb{R}^2 , then we can write it in terms of the definition above as $q_{O(2)}(\tau_{x+\mathbb{R}v}) = q_{O(2)}(\tau_x \exp(\mathbb{R}v))$.

While lines themselves are defined in terms of motion along them, we will define parallelism in terms of transformations. We start with two lines, and in order to see whether they are parallel, we shift one of those lines onto the other via a translation.

Definition 1.2. Two lines ℓ and ℓ' in the Euclidean plane are *parallel* if and only if there is some $u \in \mathbb{R}^2$ such that $\tau_u(\ell) = \ell'$.

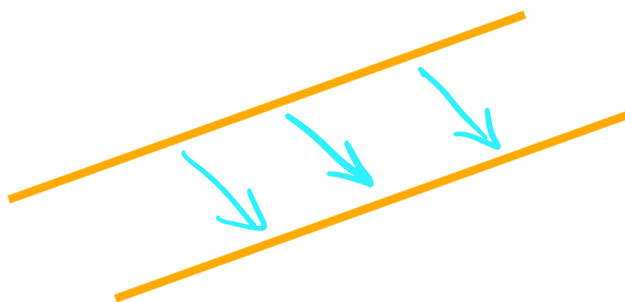


FIGURE 2. Two lines in the Euclidean plane are parallel when one is a translation of the other

Later, we will show that this definition is equivalent to the classical definition of parallel lines in terms of intersection.

Given two vectors in the same tangent space, we can measure the angle between them in terms of the rotation needed to move from one to the other.

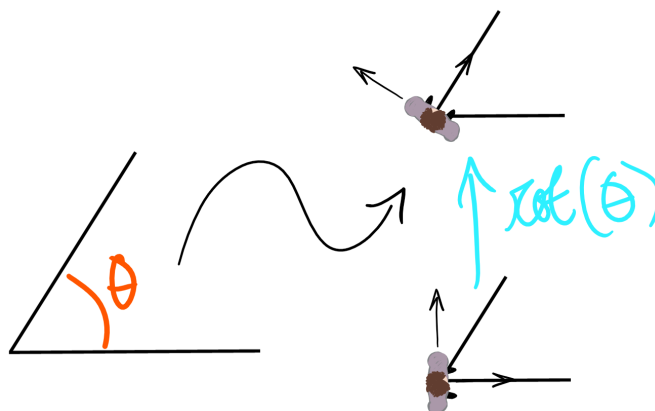


FIGURE 3. We can define angles between vectors in terms of the rotations needed to move between them

Definition 1.3. Let $g \in \mathrm{I}(2)$ be such that $q_{\mathrm{O}(2)*}((\omega_{\mathrm{I}(2)})_g^{-1}(e_1))$ is a positive scalar multiple of $v \in T_x\mathbb{R}^2$. For $w \in T_x\mathbb{R}^2$, the *(oriented) angle* from v to w is the unique $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ such that $q_{\mathrm{O}(2)*}((\omega_{\mathrm{I}(2)})_{\mathrm{rot}(\theta)}^{-1}(e_1))$ is a positive scalar multiple of w .

Note that this defines angles modulo 2π . If we want to talk about angles larger than 2π , or if we want to distinguish angles that are the same modulo 2π , then we would need to work with the universal cover \mathbb{R} of $\mathrm{SO}(2) \simeq \mathbb{R}/2\pi\mathbb{Z}$.

An interesting feature of the geometry of the Euclidean plane is that, fixing an orientation by restricting ourselves to the identity component $\mathrm{I}^\circ(2) \simeq \mathbb{R}^2 \rtimes \mathrm{SO}(2)$ of $\mathrm{I}(2)$, we have a natural way to add angles at different points. Because \mathbb{R}^2 is normal in $\mathrm{I}^\circ(2)$, we get a natural homomorphic quotient map

$$\pi_{\mathbb{R}^2} : \mathrm{I}^\circ(2) \rightarrow \mathbb{R}^2 \setminus \mathrm{I}^\circ(2) \simeq \mathrm{SO}(2)$$

given by $\tau_u \circ A \mapsto A$. Under the map $\pi_{\mathbb{R}^2}$, we can identify angles at different points on the plane and add them together.

An alternative way to describe angles comes from the adjoint representation $\mathrm{Ad} : \mathrm{I}(2) \rightarrow \mathrm{GL}(\mathfrak{i}(2))$. Conveniently, we already have some idea of what conjugation looks like, so since Ad_g is, by definition, just the pushforward at the identity of conjugation by g , we can get a fairly good picture of what the adjoint representation looks like as well.

For example, conjugating g by $\mathrm{rot}(\theta)$ gives the transformation that behaves like the motion g at the orthonormal frame corresponding to $\mathrm{rot}(\theta)$. Thus, $\mathrm{rot}(\theta) \circ \tau_{tv} \circ \mathrm{rot}(\theta)^{-1}$ is just the transformation coinciding with the motion τ_{tv} at $\mathrm{rot}(\theta)$, namely $\tau_{\mathrm{rot}(\theta) \cdot v}$, so $\mathrm{Ad}_{\mathrm{rot}(\theta)}$ just rotates velocities by θ . The angle between two vectors $v, w \in T_x\mathbb{R}^2$ can then equivalently be described as the element $\theta \in \mathbb{R}/2\pi\mathbb{Z} \simeq \mathrm{SO}(2)$ such that, for v a positive scalar multiple of $q_{\mathrm{O}(2)*}((\omega_{\mathrm{I}(2)})_g^{-1}(e_1))$, w is a positive scalar multiple of $q_{\mathrm{O}(2)*}((\omega_{\mathrm{I}(2)})_g^{-1}(\mathrm{Ad}_{\mathrm{rot}(\theta)}(e_1)))$.

Note that, in the first definition of angle above, we had v be a positive scalar multiple of $q_{\mathrm{O}(2)*}((\omega_{\mathrm{I}(2)})_g^{-1}(e_1))$ and w be a positive scalar multiple of $q_{\mathrm{O}(2)*}((\omega_{\mathrm{I}(2)})_{\mathrm{rot}(\theta)}^{-1}(e_1))$. Examining this second expression more closely, we have

$$\begin{aligned} q_{\mathrm{O}(2)*}((\omega_{\mathrm{I}(2)})_{\mathrm{rot}(\theta)}^{-1}(e_1)) &= q_{\mathrm{O}(2)*}(\mathrm{L}_{\mathrm{rot}(\theta)*}(e_1)) = q_{\mathrm{O}(2)*}(\mathrm{L}_{g*} \mathrm{L}_{\mathrm{rot}(\theta)*}(e_1)) \\ &= q_{\mathrm{O}(2)*}(\mathrm{R}_{\mathrm{rot}(\theta)^{-1}*}(\mathrm{L}_{g*} \mathrm{L}_{\mathrm{rot}(\theta)*}(e_1))) \\ &= q_{\mathrm{O}(2)*}(\mathrm{L}_{g*}(\mathrm{L}_{\mathrm{rot}(\theta)*} \mathrm{R}_{\mathrm{rot}(\theta)^{-1}*})(e_1)) \\ &= q_{\mathrm{O}(2)*}(\mathrm{L}_{g*} \mathrm{Ad}_{\mathrm{rot}(\theta)}(e_1)) \\ &= q_{\mathrm{O}(2)*}((\omega_{\mathrm{I}(2)})_g^{-1}(\mathrm{Ad}_{\mathrm{rot}(\theta)}(e_1))), \end{aligned}$$

where the equality in the second line follows from $q_{\mathrm{O}(2)} \circ \mathrm{R}_{\mathrm{rot}(\theta)} = q_{\mathrm{O}(2)}$ and the equality in the third line is a consequence of left-translation

Small correction: In order to uniquely determine the direction of the angle here, we pick both a starting vector and, implicitly, an *orientation*. Thus, these should really be called *oriented angles*, since they are not preserved by orientation-reversing isometries. We can get un-oriented angles by forgetting the direction, though this only determines an element of $\mathbb{R}/2\pi\mathbb{Z}$ up to sign, so we lose meaningful angle-addition.

and right-translation commuting with each other. This verifies that the two definitions of angle are equivalent.

Finally, we get to circles. We won't use them much in the Euclidean geometry planned for this lecture, but it's still worth giving them a definition in terms of $I(2)$, to prove that we can.

Definition 1.4. For $x \in \mathbb{R}^2$ and $v \in \mathbb{R}^2$, the *circle* centered at x with radius (the length of) v is the set

$$C_v(x) := \left\{ q_{O(2)}(g\tau_v) : g \in q_{O(2)}^{-1}(x) \right\}.$$

In other words, $C_v(x)$ is the set of all points that some orthonormal frame over x thinks are v away from x . Equivalently, if you stand over x and specify a radius v from your frame, and then you spin around “in a circle” until you get back to your original configuration, then you will have traced out a circle.

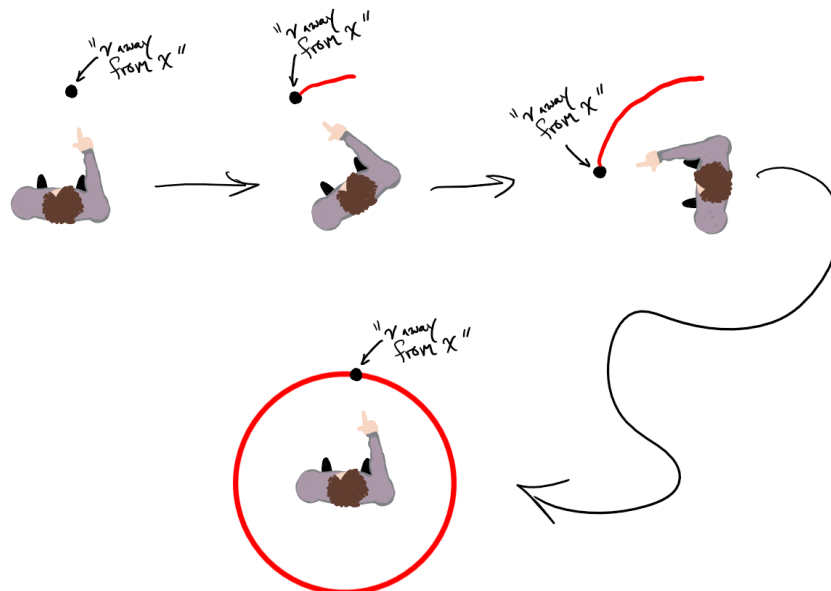


FIGURE 4. Tracing out a circle as in the above definition

2. TWO ELEMENTARY RESULTS FROM EUCLIDEAN GEOMETRY

To demonstrate how “actual” Euclidean geometry can be done in terms of isometries, we shall prove two elementary results.

Proposition 2.1. *Suppose two lines ℓ and ℓ' intersect at a point x , determining four angles around x as in Figure 5. The angles opposite each other are congruent, so that $\theta_1 = \theta_3$ and $\theta_2 = \theta_4$.*

Using $I(2)$, this is fairly straightforward: imagine you are occupying a frame g over x such that you are pointed along a vector tangent to ℓ

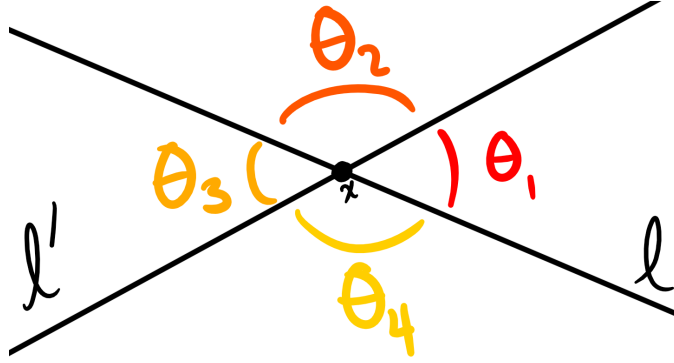


FIGURE 5. Two lines l and l' intersecting at x

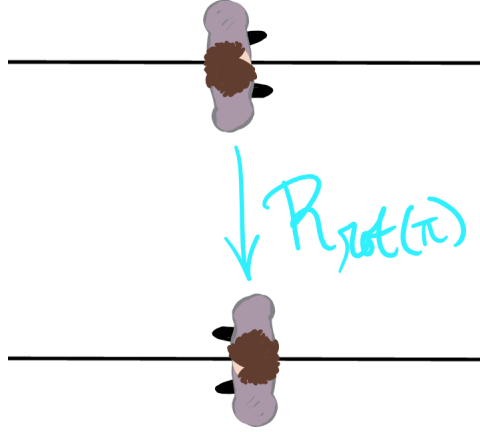


FIGURE 6. Turning on the spot by $\text{rot}(\pi) = -\mathbf{1}$ keeps you on the same line but points you in the opposite direction

used to form the angle θ_1 . By definition, if we rotate ourselves by θ_1 , which is to say we right-translate by $\text{rot}(\theta_1)$, then we will be pointing along l' . Now, imagine we are at $g \circ \text{rot}(\pi) = g \circ (-\mathbf{1})$. We are still pointed along l , but now in the opposite direction, along a vector we can use to form the angle θ_3 . But because $\text{SO}(2)$ is abelian, if we move by $\text{rot}(\theta_1)$ from $g \circ (-\mathbf{1})$, then we'll be at the orthonormal frame $g \circ (-\mathbf{1}) \circ \text{rot}(\theta_1) = (g \circ \text{rot}(\theta_1)) \circ (-\mathbf{1})$, which points us along l' again in the opposite direction as $g \circ \text{rot}(\theta_1)$. In other words, rotating by θ_1 did the same thing as rotating by θ_3 , so they are equal.

Equivalently, we could just say that θ_1 and θ_3 are congruent under the isometry $\tau_x \circ (-\mathbf{1}) \circ \tau_x^{-1}$, essentially by the same reason: $-\mathbf{1}$ sends each line through 0 to itself.

Let us try another.

Proposition 2.2. *Suppose l_1 and l'_1 are distinct parallel lines, and l_2 is a line intersecting both l_1 and l'_1 , forming the angles θ_1 and θ_2 as in Figure 7. Then, θ_1 and θ_2 are congruent.*

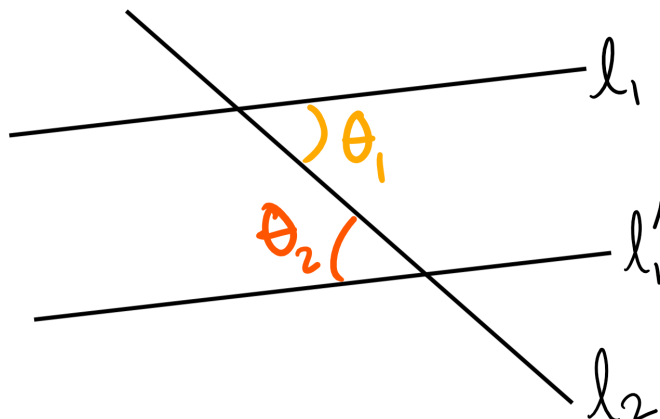


FIGURE 7. Parallel lines ℓ_1 and ℓ'_1 intersected by a transversal ℓ_2

Again, this is not too difficult in terms of $I(2)$: because ℓ_1 and ℓ'_1 are parallel, there is some $u \in \mathbb{R}^2$ such that $\tau_u(\ell_1) = \ell'_1$, and because the subgroup of translations is normal, there is some nonzero $v \in \mathbb{R}^2$ such that $\tau_v(\ell'_1) = \ell'_1$. For x the point where ℓ_1 and ℓ_2 intersect and y the point where ℓ'_1 and ℓ_2 intersect, there is some $t \in \mathbb{R}$ such that

$$\tau_u(x) = u + x = tv + y = \tau_{tv}(y),$$

hence $y - x = u - tv$, so $\tau_{y-x} = \tau_{-tv} \circ \tau_u$. Since τ_u sends ℓ_1 to ℓ'_1 and τ_{-tv} preserves ℓ'_1 , this means τ_{y-x} sends ℓ_1 to ℓ'_1 . Moreover, since $x, y \in \ell_2$, τ_{y-x} preserves ℓ_2 , so τ_{y-x} sends θ_1 to the angle opposite θ_2 in the intersection of ℓ'_1 and ℓ_2 , hence they are congruent by the first proposition.

3. HOW DOES EUCLIDEAN GEOMETRY SHOW UP, ALGEBRAICALLY?

Now that we have seen how to reformulate Euclidean geometry in terms of $I(2)$, we are led to a natural question: why $I(2)$? What about this particular Lie group gives us Euclidean geometry?

Throughout, we have relied heavily on the subgroup $\mathbb{R}^2 < I(2)$ of translations. In particular, as we saw in Proposition 2.2, we explicitly used the fact that \mathbb{R}^2 is normal in $I(2)$ to use transformations instead of motions. Implicitly, we also used the fact that $\mathbb{R}^2 < I(2)$ acts simply transitively on $I(2)/O(2) \cong \mathbb{R}^2$, when we used the points x and y to determine the transformation τ_{y-x} . Translations were also used to describe parallelism, giving a definition that, as we now show, happens to coincide with the standard formulation in terms of intersections.

Proposition 3.1. *Two distinct lines ℓ and ℓ' in the Euclidean plane are parallel if and only if $\ell \cap \ell' = \emptyset$.*

Proof. To start, choose $x, y, v, w \in \mathbb{R}^2$ such that $\ell = q_{O(2)}(\tau_{x+\mathbb{R}v})$ and $\ell' = q_{O(2)}(\tau_{y+\mathbb{R}w})$. If $\ell \cap \ell' = \emptyset$, then $x \neq y$ and there are no $t, s \in \mathbb{R}$

such that

$$q_{O(2)}(\tau_{x+tv}) = x + tv = y + sw = q_{O(2)}(\tau_{y+sw}).$$

In particular, $y - x$ is never in the span of v and w , so because \mathbb{R}^2 is 2-dimensional, this means that v and w are scalar multiples of each other and

$$\tau_{y-x}(\ell) = \tau_{y-x}(q_{O(2)}(\tau_{x+\mathbb{R}v})) = q_{O(2)}(\tau_{y+\mathbb{R}v}) = q_{O(2)}(\tau_{y+\mathbb{R}w}) = \ell'.$$

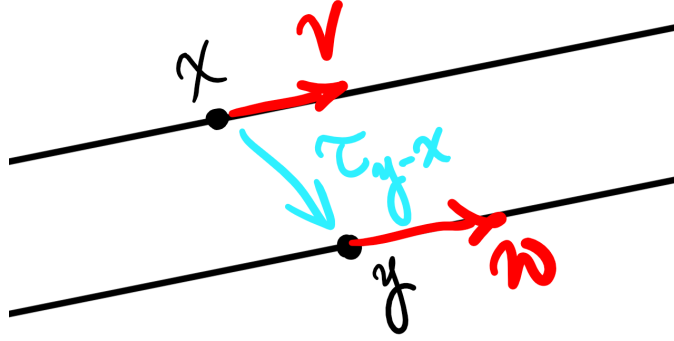


FIGURE 8. If $\ell \cap \ell' = \emptyset$, then for $x \in \ell$ and $y \in \ell'$, we have $\tau_{y-x}(\ell) = \ell'$

Conversely, if there exists $u \in \mathbb{R}^2$ such that $\tau_u(\ell) = \ell'$, then

$$\tau_u(q_{O(2)}(\tau_{x+\mathbb{R}v})) = q_{O(2)}(\tau_{u+x+\mathbb{R}v}) = q_{O(2)}(\tau_{y+\mathbb{R}w}),$$

so for every $t \in \mathbb{R}$, there is an $s \in \mathbb{R}$ such that $u + x + tv = y + sw$. In particular, $u + x - y + tv$ is in the span of w for every $t \in \mathbb{R}$, so $u + x - y$ and $(u + x - y + v) - (u + x - y) = v$ are in the span of w . Thus, if $\ell \cap \ell' \neq \emptyset$, then there would be $t, s \in \mathbb{R}$ such that $x + tw = u + x + sw$, which would mean that u is in the span of w , which would mean ℓ and ℓ' were not distinct. \square

Note that the proof of this equivalence explicitly depended on our ability to use \mathbb{R}^2 as a vector space. Moreover, it once again implicitly used the identification between \mathbb{R}^2 as the homogeneous space $I(2)/O(2)$ and \mathbb{R}^2 as the subgroup of translations. For example, we took the points x and y of the homogeneous space and used them to create the translation τ_{y-x} from their difference, and we showed that v was in the span of w using $(u + x - y + v) - (u + x - y) = v$.

Indeed, the key feature that allows for parallelism to look the way that it does in Euclidean geometry is this simply transitive normal subgroup of translations. To see this, suppose we have another Lie group G acting transitively on the plane \mathbb{R}^2 , and that G contains a closed normal subgroup isomorphic to \mathbb{R}^2 that acts simply transitively on \mathbb{R}^2 . Again, we can decompose elements $g \in G$ as $g = \tau_{g(0)}(\tau_{g(0)}^{-1}g)$,

with $\tau_{g(0)}^{-1}g$ acting linearly on \mathbb{R}^2 by conjugation (since conjugation must give an automorphism of \mathbb{R}^2 and the group of automorphisms of \mathbb{R}^2 is precisely $\text{GL}_2 \mathbb{R}$).

We can then define lines the same way we did above, in terms of translations. Our definition of parallelism in terms of translations still makes sense as well, and by repeating the proof of the above proposition, we see that it is consistent with the usual definition in terms of intersection.

Circles and angles are a bit trickier to find in the structure of $\text{I}(2)$. Of course, we can (correctly) guess that it ultimately comes from the subgroup $\text{O}(2) < \text{I}(2)$, but *how* is still a bit mysterious, since we used translations to describe both concepts. The key is to notice that we didn't actually need the translations for these definitions. For example, when defining circles, we used $v \in \mathbb{R}^2$ to describe the radius of $C_v(x)$ because that was more familiar, but really, every isometry $a \in \tau_v \text{O}(2) = q_{\text{O}(2)}^{-1}(v)$ determines the same circle:

$$C_v(x) = \left\{ q_{\text{O}(2)}(g\tau_v) : g \in q_{\text{O}(2)}^{-1}(x) \right\} = \left\{ q_{\text{O}(2)}(ga) : g \in q_{\text{O}(2)}^{-1}(x) \right\}.$$

Thus, the angles and circles of Euclidean geometry come from the stabilizer subgroup $\text{O}(2)$ of $0 \in \mathbb{R}^2 \cong \text{I}(2)/\text{O}(2)$. In particular, if we were to replace $\text{I}(2)$ with a Lie group G containing a closed subgroup isomorphic to $\text{O}(2)$, then we could use the definitions from above to talk about "circles" and "angles" in this other "geometry".

In the next lecture, we will clarify what we mean here by "geometry", and explore some famous examples.